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Estimates for Littlewood-Paley functions and extrapolation

Shuichi Sato

Abstract. We prove certain L^p-estimates for Littlewood-Paley functions arising from rough kernels. The estimates are useful for extrapolation to prove L^p-boundedness of the Littlewood-Paley functions under a sharp kernel condition.

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1. Introduction

We consider the Littlewood-Paley function on $\mathbb{R}^n$ defined by

$$S_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \, \frac{dt}{t}\right)^{1/2},$$

where ψ is in $L^1(\mathbb{R}^n)$ and $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. We always assume that

$$\int_{\mathbb{R}^n} \psi(x)\, dx = 0.$$

For the theory of Littlewood-Paley functions one can see [14], [15] and [16]. Also, recent developments can be found in [1], [4], [5], [7], [8], [9], [11], [12]. One of the well-known sufficient conditions for L^p boundedness of S_ψ is the following:

Theorem A. *Suppose that ψ satisfies*

$$|\psi(x)| \le C(1+|x|)^{-n-\epsilon} \qquad \text{for some} \quad \epsilon > 0,$$
$$\int_{\mathbb{R}^n} |\psi(x-y) - \psi(x)|\, dx \le C|y|^\epsilon \qquad \text{for some} \quad \epsilon > 0.$$

Then the operator S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

This is due to Benedek, Calderón and Panzone [3]. Fan-Sato [9] proved the following result, which substantially relaxes the conditions imposed on ψ in Theorem A.

Theorem B. *Suppose that the function ψ satisfies the following conditions*:

(1) $B_\epsilon(\psi) < \infty$ *for some* $\epsilon > 0$, *where* $B_\epsilon(\psi) = \int_{|x|\geq 1} |\psi(x)|\,|x|^\epsilon\,dx$;

(2) $D_u(\psi) < \infty$ *for some* $u > 1$, *where* $D_u(\psi) = \left(\int_{|x|\leq 1} |\psi(x)|^u\,dx\right)^{1/u}$;

(3) $|\psi(x)| \leq h(|x|)\Omega(x')$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$, *where* $x' = x/|x|$, *for some non-negative functions* h *on* $(0,\infty)$ *and* Ω *on* S^{n-1} (*the unit sphere in* $\mathbb{R}^n$) *such that*

   (a) $h(r)$ *is non-increasing on* $(0,\infty)$ *and* $h(|x|) \in L^1(\mathbb{R}^n)$,

   (b) $\Omega \in L^q(S^{n-1})$ *for some* q, $1 < q \leq \infty$.

Then S_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $p \in (1,\infty)$.

For the rest of this note we assume that ψ is compactly supported and the support is contained in the unit ball $\{|x| \leq 1\}$, for the sake of simplicity. We shall prove L^p estimates for S_ψ that are useful in extrapolation arguments to obtain a minimum condition on ψ for L^p boundedness of S_ψ.

Theorem 1. *Suppose that* $|\psi(x)| \leq h(|x|)\Omega(x')$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$, *where* h *is a non-negative, non-increasing function on* $(0,\infty)$ *with support in* $(0,1]$ *and* Ω *is a non-negative function on* S^{n-1}. *We assume that* $h(|x|) \in L^1(\mathbb{R}^n)$, $\Omega \in L^1(S^{n-1})$ *and* $\psi \in L^q(\mathbb{R}^n)$ *for some* $q > 1$. *Put* $m_\psi(x) = h(|x|)\Omega(x')$. *Then, we have*

$$\|S_\psi(f)\|_p \leq C_p\,(q/(q-1))^{1/2}\,(\|\psi\|_q + \|m_\psi\|_1)\,\|f\|_p$$

for all $p \in (1,\infty)$, *where the constant* C_p *is independent of* q, ψ, h *and* Ω; *and* $q/(q-1) = 1$ *when* $q = \infty$.

Theorem 2. *Suppose that* $\psi \in L^q(\mathbb{R}^n)$ *for some* $q > 1$. *Then, we have*

$$\|S_\psi(f)\|_p \leq C_p\,(q/(q-1))^{1/2}\,\|\psi\|_q\|f\|_p$$

for all $p \in [2,\infty)$ *with a constant* C_p *independent of* q *and* ψ.

We are interested in the L^p estimates for $S_\psi(f)$ of Theorems 1 and 2 when q is near 1.

Let $L(\log L)^{1/2}(S^{n-1})$ be the class of the functions Ω on S^{n-1} satisfying

$$\int_{S^{n-1}} |\Omega(\theta)|\,[\log(2+|\Omega(\theta)|)]^{1/2}\,d\sigma(\theta) < \infty,$$

where $d\sigma$ denotes the Lebesgue surface measure on S^{n-1}. The class $L(\log L)^{1/2}(\mathbb{R}^n)$ of functions on $\mathbb{R}^n$ is defined similarly. Using Theorems 1 and 2 and applying extrapolation, we can prove the following two results.

Corollary 1. *Let* $\Omega \in L(\log L)^{1/2}(S^{n-1})$ *and* $\Omega \geq 0$. *Suppose that* $|\psi(x)| \leq h(|x|)\Omega(x')$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$, *where* h *is as in Theorem* 1. *We further assume that* $h(|x|) \in L^q(\mathbb{R}^n)$ *for some* $q > 1$. *Then,* S_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $p \in (1,\infty)$.

Corollary 2. *Suppose that ψ is in $L(\log L)^{1/2}(\mathbb{R}^n)$. Then S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in [2, \infty)$.*

When $\psi \in L^q(\mathbb{R}^n)$ for some $q > 1$, L^p boundedness of S_ψ for $p \in [2, \infty)$ was proved in [9]. Corollary 2 is an improvement over the result. See [4] and also Remark 2 of [9] for L^p boundedness of S_ψ for $p < 2$.

Let $\psi(x) = |1 - |x|^2|^{\alpha-1}\Omega(x')$ if $0 < |x| < 1$, and $\psi(x) = 0$ if $|x| > 1$, where we assume that $\alpha > 0$, $\Omega \in L(\log L)^{1/2}(S^{n-1})$ and $\int_{S^{n-1}} \Omega \, d\sigma = 0$. Then, we can see that Corollary 2 applies to the operator S_ψ.

Let

$$\psi(x) = |x|^{-n+1}\Omega(x')\chi_{(0,1]}(|x|) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\},$$

where $\Omega \in L^1(S^{n-1})$, $\int_{S^{n-1}} \Omega \, d\sigma = 0$. Here χ_E denotes the characteristic function of a set E. Then, the Littlewood-Paley function $S_\psi(f)$ coincides with the Marcinkiewicz integral $\mu_\Omega(f)$ in Stein [14] (see also Hörmander [10]). As an application of Theorem 1 we have the following result:

Corollary 3. *If $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then*

$$\|\mu_\Omega(f)\|_p \le C_p \left(q/(q-1)\right)^{1/2} \|\Omega\|_q \|f\|_p$$

for $p \in (1, \infty)$, where the constant C_p is independent of q and Ω. Here $\|\Omega\|_q$ denotes the norm of Ω in $L^q(S^{n-1})$.

Also, as a consequence of Corollary 1, we have the following result of Al-Salman, Al-Qassem, Cheng and Pan [1].

Theorem C. *If $\Omega \in L(\log L)^{1/2}(S^{n-1})$, then μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.*

The case $p = 2$ of Theorem C is due to Walsh [17]. In Section 2 we shall prove Theorem 1. We adapt the method of [6] involving the Littlewood-Paley theory for the case of square function operators. We apply a Littlewood-Paley decomposition adapted to a suitable lacunary sequence depending on q for which $\psi \in L^q(\mathbb{R}^n)$. See [2] and also [13] for the method of appropriately choosing a lacunary sequence. Theorem 2 is proved in Section 3 by applying the Littlewood-Paley theory in like manner. In proving Theorems 1 and 2, basic estimates and key observations of [11], [9] will be used. Also, to prove Theorem 2, we apply an induction argument similar to the one used in [13] to get sharp L^p estimates for singular Radon transforms.

Applying an extrapolation method (see, e.g., Zygmund [18, Chap. XII, pp. 119–120]), we shall prove Corollary 1 by Theorem 1 in Section 4. Similarly, Corollary 2 follows from Theorem 2. Throughout this note, the letter C will be used to denote non-negative constants which may be different in different occurrences.

2. Proof of Theorem 1

For $k \in \mathbb{Z}$ (the set of integers) and $\rho \geq 2$, let T_k be an operator mapping functions on $\mathbb{R}^n$ to $\mathcal{H}$-valued functions on $\mathbb{R}^n$ defined as

$$(T_k(f)(x))(t) = (\psi_t * f)(x)\chi_{[1,\rho)}(\rho^{-k}t),$$

where $\mathcal{H}$ is the Hilbert space $L^2((0,\infty), dt/t)$. Note that

$$|T_k(f)(x)|_{\mathcal{H}} = \left(\int_{\rho^k}^{\rho^{k+1}} |\psi_t * f(x)|^2 \frac{dt}{t}\right)^{1/2}.$$

Lemma 1. *Let ψ be as in Theorem 1. Then we have*

$$\left\|\left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2\right)^{1/2}\right\|_s \leq C_s\|\psi\|_1^{1/2}\|m_\psi\|_1^{1/2}(\log\rho)^{1/2}\left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_s$$

for all $s \in (1,\infty)$, where m_ψ is as in Theorem 1 and the constant C_s is independent of ρ, ψ, h and Ω.

Proof. Define a maximal function

$$M_\psi(f)(x) = \sup_{t>0} |\psi_t * f(x)|.$$

Since the method of rotations implies $\|M_\psi(f)\|_r \leq C_r\|m_\psi\|_1\|f\|_r$ for all $r > 1$, arguing as in the proofs of Lemmas 1 and 2 of [9] and checking the constants in the arguments, we can obtain Lemma 1. □

Lemma 2. *Let ψ be as in Theorem 2. Then*

$$\int_{\rho^k}^{\rho^{k+1}} \left|\hat{\psi}(t\xi)\right|^2 \frac{dt}{t} \leq C(\log\rho)\|\psi\|_q^2 \min(1, |\rho^{k+1}\xi|, |\rho^k\xi|^{-1})^{1/(2q')},$$

where the constant C is independent of $\rho \geq 2$, $q > 1$ and ψ. Here $q' = q/(q-1)$, and $\hat{g}$ denotes the Fourier transform of g; we also write $\hat{g} = \mathcal{F}(g)$.

Proof. First, a direct computation implies that

$$\int_1^\rho \left|\hat{\psi}(t\rho^k\xi)\right|^2 dt/t \leq (\log\rho)\|\psi\|_1^2 \leq C(\log\rho)\|\psi\|_q^2. \tag{2.1}$$

Next, using Lemmas 2 and 3 of [11], we see that

$$\begin{aligned}\int_1^\rho \left|\hat{\psi}(t\rho^k\xi)\right|^2 dt/t &\leq \sum_{0\leq m\leq(\log\rho)/\log 2} \int_1^2 \left|\hat{\psi}(t2^m\rho^k\xi)\right|^2 dt/t \\ &\leq \sum_{0\leq m\leq(\log\rho)/\log 2} C\|\psi\|_q^2 \left|2^m\rho^k\xi\right|^{-1/(2q')} \qquad (2.2)\\ &\leq C(\log\rho)\|\psi\|_q^2 \left|\rho^k\xi\right|^{-1/(2q')}.\end{aligned}$$

Finally, by the proof of Lemma 1 of [11], we have $|\hat{\psi}(\xi)| \le C|\xi|\|\psi\|_1$, and hence

$$\begin{aligned}\int_1^\rho \left|\hat{\psi}(t\rho^k\xi)\right|^2 dt/t &\le C\|\psi\|_1^2 \int_1^\rho |t\rho^k\xi|^2\, dt/t \le C\|\psi\|_1^2|\rho^{k+1}\xi|^2 \\ &\le C(\log\rho)\|\psi\|_q^2|\rho^{k+1}\xi|^2.\end{aligned} \tag{2.3}$$

Combining (2.1)–(2.3), we have the conclusion, since

$$\min(1, |\rho^{k+1}\xi|^2, |\rho^k\xi|^{-1/(2q')}) \le \min(1, |\rho^{k+1}\xi|, |\rho^k\xi|^{-1})^{1/(2q')}.$$

□

We choose a sequence $\{\Psi_k\}_{-\infty}^{\infty}$ of non-negative functions in $C^\infty(\mathbb{R})$ such that

$$\operatorname{supp}(\Psi_k) \subset [\rho^{-k-1}, \rho^{-k+1}], \quad \sum_k \Psi_k(t) = 1 \quad \text{for } t > 0.$$

We further assume that

$$|(d/dt)^j\Psi_k(t)| \le c_j|t|^{-j} \quad (j = 1, 2, \dots),$$

where the constants c_j are independent of ρ and k. This is feasible since $\rho \ge 2$. Define Fourier multiplier operators Δ_j as

$$\mathcal{F}(\Delta_j(f))(\xi) = \Psi_j(|\xi|)\hat{f}(\xi) \quad \text{for } j \in \mathbb{Z}$$

and decompose $(\psi_t * f)(x)$ as $(\psi_t * f)(x) = \sum_{j\in\mathbb{Z}} F_j(x,t)$, where

$$F_j(x,t) = \sum_{k\in\mathbb{Z}} \Delta_{j+k}(\psi_t * f)(x)\chi_{[\rho^k,\rho^{k+1})}(t).$$

Then, we have $S_\psi(f)(x) \le \sum_{j\in\mathbb{Z}} U_j(f)(x)$, where

$$U_j(f)(x) = \left(\int_0^\infty |F_j(x,t)|^2 \frac{dt}{t}\right)^{1/2} = \left(\sum_k |T_k(\Delta_{j+k}(f))(x)|_{\mathcal{H}}^2\right)^{1/2}.$$

Let $E_j = \{\rho^{-1-j} \le |\xi| \le \rho^{1-j}\}$. Then by the Plancherel theorem and Lemma 2 we have

$$\begin{aligned}\|U_j(f)\|_2^2 &= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} \int_{\rho^k}^{\rho^{k+1}} |\Delta_{j+k}(\psi_t * f)(x)|^2 \frac{dt}{t}\, dx \\ &\le \sum_{k\in\mathbb{Z}} C \int_{E_{j+k}} \left(\int_{\rho^k}^{\rho^{k+1}} \left|\hat{\psi}(t\xi)\right|^2 \frac{dt}{t}\right) \left|\hat{f}(\xi)\right|^2 d\xi \\ &\le \sum_{k\in\mathbb{Z}} C(\log\rho)\|\psi\|_q^2 \int_{E_{j+k}} \min(1, |\rho^{k+1}\xi|, |\rho^k\xi|^{-1})^{1/(2q')} \left|\hat{f}(\xi)\right|^2 d\xi \\ &\le C(\log\rho)\|\psi\|_q^2 \min(1, \rho^{-|j|+2})^{1/(2q')} \sum_{k\in\mathbb{Z}} \int_{E_{j+k}} \left|\hat{f}(\xi)\right|^2 d\xi \\ &\le C(\log\rho)\|\psi\|_q^2 \min(1, \rho^{-|j|+2})^{1/(2q')} \|f\|_2^2,\end{aligned}$$

where the last inequality is valid since the sets E_j are finitely overlapping. Thus we have

$$\|U_j(f)\|_2 \le C(\log\rho)^{1/2}\|\psi\|_q \min(1,\rho^{-|j|+2})^{1/(4q')}\|f\|_2. \tag{2.4}$$

We now use the Littlewood-Paley inequality

$$\left\|\left(\sum_k |\Delta_k(f)|^2\right)^{1/2}\right\|_p \le c_p\|f\|_p, \tag{2.5}$$

where $1 < p < \infty$ and the constant c_p is independent of ρ. By (2.5) and Lemma 1 we see that

$$\begin{aligned}\|U_j(f)\|_s &= \left\|\left(\sum_{k\in\mathbb{Z}} |T_k(\Delta_{j+k}(f))|_{\mathcal{H}}^2\right)^{1/2}\right\|_s \\ &\le C\|\psi\|_1^{1/2}\|m_\psi\|_1^{1/2}(\log\rho)^{1/2}\left\|\left(\sum_{k\in\mathbb{Z}} |\Delta_{j+k}(f)|^2\right)^{1/2}\right\|_s \\ &\le C\|\psi\|_1^{1/2}\|m_\psi\|_1^{1/2}(\log\rho)^{1/2}\|f\|_s\end{aligned} \tag{2.6}$$

for $s \in (1,\infty)$. Interpolating between (2.4) and (2.6), we get

$$\begin{aligned}\|U_j(f)\|_p &\le C(\log\rho)^{1/2}\left(\|\psi\|_q \min(1,\rho^{-|j|+2})^{1/(4q')}\right)^\eta \left(\|\psi\|_1^{1/2}\|m_\psi\|_1^{1/2}\right)^{1-\eta}\|f\|_p \\ &\le C(\log\rho)^{1/2}\|\psi\|_q^{(1+\eta)/2}\|m_\psi\|_1^{(1-\eta)/2}\min(1,\rho^{-|j|+2})^{\eta/(4q')}\|f\|_p\end{aligned}$$

for some $\eta \in (0,1]$ depending on p, where $1 < p < \infty$. Thus

$$\begin{aligned}\|S_\psi(f)\|_p &\le \sum_{j\in\mathbb{Z}}\|U_j(f)\|_p \\ &\le C\left(\sum_{j\in\mathbb{Z}}\min(1,\rho^{-|j|+2})^{\eta/(4q')}\right)(\log\rho)^{1/2}\|\psi\|_q^{(1+\eta)/2}\|m_\psi\|_1^{(1-\eta)/2}\|f\|_p \\ &\le C(1-\rho^{-\eta/(4q')})^{-1}(\log\rho)^{1/2}\|\psi\|_q^{(1+\eta)/2}\|m_\psi\|_1^{(1-\eta)/2}\|f\|_p \\ &\le C(1-\rho^{-\eta/(4q')})^{-1}(\log\rho)^{1/2}\left(\|\psi\|_q + \|m_\psi\|_1\right)\|f\|_p,\end{aligned}$$

where the last inequality follows from Young's inequality. Taking $\rho = 2^{q'}$, we get the conclusion of Theorem 1, since

$$(1-\rho^{-\eta/(4q')})^{-1}(\log\rho)^{1/2} = (1-2^{-\eta/4})^{-1}\left((q\log 2)/(q-1)\right)^{1/2}$$

if $\rho = 2^{q'}$.

3. Proof of Theorem 2

Let ψ be as in Theorem 2 and $\rho \geq 2$.

Lemma 3. *Let $\theta \in (0,1)$. Suppose $2 \leq s < 2(1+\theta)/\theta$ and put $r = (s/2)'$. Then we have*

$$\left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_s \leq C_s (\log \rho)^{1/2} \|\psi\|_q \left(1 - \rho^{-\theta/(4q')}\right)^{-1/r} \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s,$$

where the constant C_s is independent of ψ, $q > 1$ and ρ.

To prove this we need the following:

Lemma 4. *Let $p(x) = \int_1^\rho |\psi_t(x)|\, dt/t$. Define a maximal function $N_\psi^{(\rho)}(f)(x) = \sup_{k \in \mathbb{Z}} |(f * p_{\rho^k})(x)|$. Let $\theta \in (0,1)$. Then, for $p > 1 + \theta$ we have*

$$\|N_\psi^{(\rho)}(f)\|_p \leq C_p (\log \rho) \|\psi\|_q \left(1 - \rho^{-\theta/(4q')}\right)^{-2/p} \|f\|_p,$$

where the constant C_p is independent of ψ, $q > 1$ and ρ.

Proof. Let $\varphi \in C^\infty(\mathbb{R}^n)$ be compactly supported and satisfy $\hat{\varphi}(0) = 1$, $\varphi \geq 0$. Define $\Phi(x) = p(x) - (\log \rho)\|\psi\|_1 \varphi(x)$. Then

$$\|\Phi_{\rho^k}\|_1 \leq C (\log \rho) \|\psi\|_q, \tag{3.1}$$

$$\left|\mathcal{F}(\Phi_{\rho^k})(\xi)\right| \leq C (\log \rho) \|\psi\|_q \min\left(|\rho^{k+1}\xi|, |\rho^k \xi|^{-1}\right)^{1/(4q')}. \tag{3.2}$$

Note that $\|p\|_1 = (\log \rho)\|\psi\|_1$ and

$$|\hat{p}(\xi)| \leq (\log \rho)^{1/2} \left(\int_1^\rho |\mathcal{F}(|\psi|)(t\xi)|^2\, dt/t \right)^{1/2}.$$

Using these observations and arguing as in the proof of Lemma 2, we can prove the estimates (3.1), (3.2).

We see that $\|N_\psi^{(\rho)}(f)\|_\infty \leq C(\log \rho)\|\psi\|_q \|f\|_\infty$, since $\|p_{\rho^k}\|_1 \leq C(\log \rho)\|\psi\|_q$. So, Lemma 4 follows by interpolation if we prove it for $p \in (1+\theta, 2]$. Define

$$g(f)(x) = \left(\sum_{k=-\infty}^{\infty} \left|\Phi_{\rho^k} * f(x)\right|^2 \right)^{1/2}.$$

Then

$$\begin{aligned} N_\psi^{(\rho)}(f) &\leq g(f) + (\log \rho)\|\psi\|_1 \sup_{t>0} |\varphi_t * f| \\ &\leq g(f) + C(\log \rho)\|\psi\|_1 M(f), \end{aligned} \tag{3.3}$$

where $M(f)$ denotes the Hardy-Littlewood maximal function. Thus, it suffices to prove $\|g(f)\|_p \leq CAB^{2/p}\|f\|_p$ for $p \in (1+\theta, 2]$, where $A = (\log \rho)\|\psi\|_q$ and $B =$

$\left(1-\rho^{-\theta/(4q')}\right)^{-1}$. Recalling a well-known property of Rademacher's functions, we can see that this follows from

$$\|V_\epsilon(f)\|_p \le CAB^{2/p}\|f\|_p \tag{3.4}$$

for $p \in (1+\theta, 2]$ with a constant C independent of ϵ, where $V_\epsilon(f) = \sum_k \epsilon_k \Phi_{\rho^k} * f$, $\epsilon = \{\epsilon_k\}$, $\epsilon_k = 1$ or -1.

To prove (3.4) we need the following.

Lemma 5. *We define a sequence $\{p_j\}_1^\infty$ by $p_1 = 2$ and $1/p_{j+1} = 1/2 + (1-\theta)/(2p_j)$ for $j \ge 1$. Then, for $j \ge 1$ we have*

$$\|V_\epsilon(f)\|_{p_j} \le C_j AB^{2/p_j} \|f\|_{p_j}, \tag{3.5}$$

where C_j is independent of ψ, $q > 1$, ρ and ϵ.

We note that $\{p_j\}$ is decreasing and converges to $1+\theta$.

Proof of Lemma 5. Let

$$V_j(f) = \sum_{k=-\infty}^{\infty} \epsilon_k \left(\Phi_{\rho^k} * \Delta_{j+k}(f)\right).$$

where the operators Δ_j are as in the proof of Theorem 1. Then, $V_\epsilon(f) = \sum_j V_j(f)$. Using Plancherel's theorem and the estimates (3.1), (3.2) as in the proof of (2.4), we have

$$\|V_j(f)\|_2 \le CA \min\left(1, \rho^{-|j|+2}\right)^{1/(4q')} \|f\|_2. \tag{3.6}$$

It follows that $\|V_\epsilon(f)\|_2 \le \sum_j \|V_j(f)\|_2 \le CAB\|f\|_2$. This proves (3.5) for $j = 1$.

We now assume (3.5) for $j = s$ and prove it for $j = s+1$. It will complete the proof of Lemma 5 by induction. First, by (3.3) we see that

$$\begin{aligned}
\sup_k \left||\Phi_{\rho^k}| * f\right| &\le \sup_k \left(p_{\rho^k} * |f|\right) + (\log\rho)\|\psi\|_1 \sup_{t>0} (\varphi_t * |f|) \\
&\le g(|f|) + 2(\log\rho)\|\psi\|_1 \sup_{t>0} (\varphi_t * |f|) \\
&\le g(|f|) + C(\log\rho)\|\psi\|_1 M(f).
\end{aligned}$$

From this we see that

$$\left\|\sup_k \left||\Phi_{\rho^k}| * f\right|\right\|_{p_s} \le CAB^{2/p_s}\|f\|_{p_s}, \tag{3.7}$$

since our assumption implies $\|g(f)\|_{p_s} \le CAB^{2/p_s}\|f\|_{p_s}$.

Let $1/v - 1/2 = 1/(2p_s)$. Then, by the proof of Lemma in [6, p. 544] and the estimates (3.1), (3.7), we have the vector valued inequality

$$\left\|\left(\sum_k |\Phi_{\rho^k} * g_k|^2\right)^{1/2}\right\|_v \le CAB^{1/p_s} \left\|\left(\sum_k |g_k|^2\right)^{1/2}\right\|_v. \tag{3.8}$$

For the sake of completeness, here we give a proof of (3.8). By duality it suffices to prove (3.8) with v' in place of v. Note that $p_s = (v'/2)'$. Take a non-negative $u \in L^{p_s}$ such that $\|u\|_{p_s} \le 1$ and

$$I = \int \left(\sum_k |\Phi_{\rho^k} * g_k|^2 \right) u \, dx,$$

where

$$I = \left\| \left(\sum_k |\Phi_{\rho^k} * g_k|^2 \right)^{1/2} \right\|_{v'}^2 .$$

Since $|\Phi_{\rho^k} * g_k|^2 \le \|\Phi\|_1 \left(|\Phi_{\rho^k}| * |g_k|^2 \right)$,

$$I \le \|\Phi\|_1 \sum_k \int \left(|\Phi_{\rho^k}| * |g_k|^2 \right) u \, dx \le \|\Phi\|_1 \sum_k \int |g_k|^2 \sup_k |\tilde{\Phi}_{\rho^k}| * u \, dx,$$

where $\tilde{\Phi}(x) = \Phi(-x)$. Thus, applying Hölder's inequality, by (3.1), (3.7) we have

$$I \le \|\Phi\|_1 \left\| \left(\sum_k |g_k|^2 \right)^{1/2} \right\|_{v'}^2 \left\| \sup_k |\tilde{\Phi}_{\rho^k}| * u \right\|_{p_s} \le C A^2 B^{2/p_s} \left\| \left(\sum_k |g_k|^2 \right)^{1/2} \right\|_{v'}^2$$

as claimed.

Also, from the Littlewood-Paley theory it follows that

$$\|V_j(f)\|_p \le c_p \left\| \left(\sum_k |\Phi_{\rho^k} * \Delta_{j+k}(f)|^2 \right)^{1/2} \right\|_p , \tag{3.9}$$

where $1 < p < \infty$ and c_p is independent of ρ. By (2.5), (3.8) and (3.9) we have

$$\|V_j(f)\|_v \le C A B^{1/p_s} \|f\|_v. \tag{3.10}$$

Since $1/p_{s+1} = \theta/2 + (1-\theta)/v$, interpolating between (3.6) and (3.10), we have

$$\|V_j(f)\|_{p_{s+1}} \le C A B^{(1-\theta)/p_s} \min\left(1, \rho^{-\theta(|j|-2)/(4q')}\right) \|f\|_{p_{s+1}}.$$

Therefore,

$$\begin{aligned} \|V_\epsilon(f)\|_{p_{s+1}} &\le \sum_j \|V_j(f)\|_{p_{s+1}} \le C A B^{(1-\theta)/p_s} (1 - \rho^{-\theta/(4q')})^{-1} \|f\|_{p_{s+1}} \\ &\le C A B^{2/p_{s+1}} \|f\|_{p_{s+1}}. \end{aligned}$$

This proves (3.5) for $j = s+1$, which completes the proof of Lemma 5. □

We now prove (3.4) for $p \in (1+\theta, 2]$. Let $\{p_j\}_1^\infty$ be as in Lemma 5. Then we have $p_{N+1} < p \le p_N$ for some N. Thus, interpolating between the estimates in (3.5) for $j = N$ and $j = N+1$, we have (3.4). This completes the proof of Lemma 4.

□

Proof of Lemma 3. From the proof of Lemma 1 in [9], we see that

$$\left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_s^2 \leq C\|\psi\|_1 \sum_k \int |f_k|^2 \tilde{N}_\psi^{(\rho)}(g)\, dx, \tag{3.11}$$

where $\tilde{N}_\psi^{(\rho)}(f) = \sup_{k\in\mathbb{Z}} \left|\tilde{p}_{\rho^k} * f\right|$ with $\tilde{p}(x) = p(-x)$, and g is a non-negative function in L^r satisfying $\|g\|_r \leq 1$, $r = (s/2)'$. The range of s in Lemma 3 implies that $r > 1 + \theta$. Therefore, by Hölder's inequality and Lemma 4 it follows that

$$\begin{aligned}\sum_k \int |f_k|^2 \tilde{N}_\psi^{(\rho)}(g)\, dx &\leq C \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s^2 \|\tilde{N}_\psi^{(\rho)}(g)\|_r \\ &\leq C(\log\rho)\|\psi\|_q \left(1 - \rho^{-\theta/(4q')}\right)^{-2/r} \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s^2 .\end{aligned}$$

Combining this with (3.11), we can get the conclusion of Lemma 3. □

Now we can give a proof of Theorem 2. Given $p \in [2, \infty)$, let s, $\theta \in (0, 1)$ be such that $2 \leq p < s < 2(1 + \theta)/\theta$. Let U_j be as in the proof of Theorem 1. Then, by Lemma 3 and (2.5) we see that

$$\|U_j(f)\|_s \leq C(\log\rho)^{1/2}\|\psi\|_q \left(1 - \rho^{-\theta/(4q')}\right)^{-1/r} \|f\|_s,$$

where $r = (s/2)'$. Interpolating between this estimate and (2.4), we have

$$\|U_j(f)\|_p \leq C(\log\rho)^{1/2}\|\psi\|_q \min(1, \rho^{-|j|+2})^{\eta/(4q')} \left(1 - \rho^{-\theta/(4q')}\right)^{-(1-\eta)/r} \|f\|_p$$

for some $\eta \in (0, 1]$ (we note that (2.4) holds under the assumptions of ψ in Theorem 2). Thus, arguing as in the proof of Theorem 1, we get

$$\|S_\psi(f)\|_p \leq C(\log\rho)^{1/2}\|\psi\|_q \left(1 - \rho^{-\eta/(4q')}\right)^{-1} \left(1 - \rho^{-\theta/(4q')}\right)^{-(1-\eta)/r} \|f\|_p.$$

If we put $\rho = 2^{q'}$, we get the desired estimate in Theorem 2.

4. Proof of Corollary 1

Fix $p \in (1, \infty)$ and f with $\|f\|_p \leq 1$. We write $H(x) = h(|x|)$, where h is as in Corollary 1. Put $R(\psi) = \|S_\psi(f)\|_p$. Let $F_k = \{\theta \in S^{n-1} : 2^{k-1} < |\Omega(\theta)| \leq 2^k\}$ for $k \geq 2$ and $F_1 = \{\theta \in S^{n-1} : |\Omega(\theta)| \leq 2\}$. Let $\Omega_k(\theta) = \Omega(\theta)\chi_{F_k}(\theta)$ for $k \geq 1$. We define $E_k = \{x \in B(0, 1) : x \neq 0, x' \in F_k\}$ for $k = 1, 2, 3, \ldots$, where $B(0, 1) = \{|x| \leq 1\}$. We decompose ψ as $\psi = \sum_{k=1}^\infty \psi_k$, where

$$\psi_k = \psi\chi_{E_k} - |B(0, 1)|^{-1} \left(\int_{E_k} \psi\, dx \right) \chi_{B(0,1)}.$$

We note that $\int \psi_k \, dx = 0$ and $|\psi_k(x)| \le h^*(|x|)\Omega_k^*(x')$ for $x \in \mathbb{R}^n \setminus \{0\}$, where

$$h^*(|x|) = [h(|x|) + C\|H\|_1]\,\chi_{(0,1]}(|x|), \quad \Omega_k^*(x') = \Omega_k(x') + \|\Omega_k\|_1.$$

We see that $\|\Omega_k^*\|_r \le C2^k e_k^{1/r}$ for $1 < r < \infty$, where $e_k = \sigma(F_k)$, for $k \ge 1$, and that $\|H^*\|_q \le C\|H\|_q$, where $H^*(x) = h^*(|x|)$.

Let $m_{\psi_k}(x) = h^*(|x|)\Omega_k^*(x')$ for $k \ge 1$. Then, applying Theorem 1 and using subadditivity of $R(\psi)$, we have

$$\begin{aligned}
\|S_\psi(f)\|_p = R(\psi) &\le \sum_{k\ge1} R(\psi_k) \\
&\le C(q/(q-1))^{1/2} \sum_{k<1/(q-1)} \|m_{\psi_k}\|_q + C \sum_{k\ge1/(q-1)} k^{1/2}\|m_{\psi_k}\|_{1+1/k} \\
&\le C_q\|H\|_q + C\|H\|_q \sum_{k\ge1/(q-1)} k^{1/2}\|\Omega_k^*\|_{1+1/k} \\
&\le C_q\|H\|_q + C\|H\|_q \sum_{k\ge1/(q-1)} k^{1/2}2^k e_k^{k/(k+1)}.
\end{aligned}$$

Using Young's inequality, we can see that the sum $\sum_{k\ge1} k^{1/2}2^k e_k^{k/(k+1)}$ is bounded by

$$\begin{aligned}
&2\sum_{k\ge1}(k/(k+1))\left(k^{(1+1/k)/2}2^{k(1+1/k)}e_k\right) + 2\sum_{k\ge1} 2^{-k-1}/(k+1) \\
&\le C\sum_{k\ge1} k^{1/2}2^k e_k + C \le C\int_{S^{n-1}} |\Omega(\theta)|\left(\log(2+|\Omega(\theta)|)\right)^{1/2}\, d\sigma(\theta) + C.
\end{aligned}$$

Combining results, we get the conclusion of Corollary 1.

References

[1] A. Al-Salman, H. Al-Qassem, L. C. Cheng and Y. Pan, *L^p bounds for the function of Marcinkiewicz*, Math. Res. Lett. **9** (2002), 697–700.

[2] A. Al-Salman and Y. Pan, *Singular integrals with rough kernels in $L\log L(S^{n-1})$*, J. London Math. Soc. (2) **66** (2002), 153–174.

[3] A. Benedek, A. P. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U. S. A. **48** (1962), 356–365.

[4] L. C. Cheng, *On Littlewood-Paley functions*, Proc. Amer. Math. Soc. **135** (2007), 3241–3247.

[5] Y. Ding, D. Fan and Y. Pan, *Weighted boundedness for a class of rough Marcinkiewicz integrals*, Indiana Univ. Math. J. **48** (1999), 1037–1055.

[6] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), 541–561.

[7] J. Duoandikoetxea and E. Seijo, *Weighted inequalities for rough square functions through extrapolation*, Studia Math. **149** (2002), 239–252.

[8] D. Fan and S. Sato, *Weak type* $(1,1)$ *estimates for Marcinkiewicz integrals with rough kernels*, Tohoku Math. J. **53** (2001), 265–284.

[9] D. Fan and S. Sato, *Remarks on Littlewood-Paley functions and singular integrals*, J. Math. Soc. Japan **54** (2002), 565–585.

[10] L. Hörmander, *Estimates for translation invariant operators in* L^p *spaces*, Acta Math. **104** (1960), 93–139.

[11] S. Sato, *Remarks on square functions in the Littlewood-Paley theory*, Bull. Austral. Math. Soc. **58** (1998), 199–211.

[12] S. Sato, *Singular integrals and Littlewood-Paley functions*, Selected papers on differential equations and analysis, Translations. Series 2. **215** (2005), 57-78, American Mathematical Society, Providence, RI.

[13] S. Sato, *Estimates for singular integrals and extrapolation*, arXiv:0704.1537v1 [math.CA].

[14] E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.

[15] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

[16] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, 1993.

[17] T. Walsh, *On the function of Marcinkiewicz*, Studia Math. **44** (1972), 203–217.

[18] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, Cambridge, London, New York and Melbourne, 1977.

Shuichi Sato
Department of Mathematics
Faculty of Education
Kanazawa University
Kanazawa 920-1192
Japan
e-mail: `shuichi@kenroku.kanazawa-u.ac.jp`